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A NOTE ON SCHLÄFLI TYPE MODULAR EQUATIONS OF COMPOSITE DEGREES

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Abstract. S. Ramanujan documented certain modular equations in his recordings and many mathematicians employed them for the explicit calculation of theta functions, Weber-class invariants, Continueed fractions and many more. Motivated by their work, in this paper, we found few modular equations of the similar type.

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1. Introduction, Notations and Definitions

S. Ramanujan in his notebook [7][5, pp. 204-237], documented many modular equations. For example,
let

$$P = \frac{f(-q)}{q^{1/6}f(-q^5)} \qquad \text{and} \qquad Q = \frac{f(-q^2)}{q^{1/3}f(-q^{10})}$$

then

$$PQ + \frac{5}{PQ} = \left(\frac{P}{Q}\right)^3 + \left(\frac{Q}{P}\right)^3,$$

where

$$f(-q) = (q;q)_\infty = \prod_{n=1}^{\infty}(1-q^n), \qquad |q|<1.$$

After the publication of [5], many Mathematicians, developed the modular equation of the above type and employed them for the evaluation of theta function, Weber-class invariants, continued fractions and many more. For the wonderful work one may refer [1, 2, 3, 8, 9, 11, 12]. Motivated by the above work, in this paper, we obtain modular equations of the similar type. All through the article, we shall employ the classic q-notation. The q-shifted factorial for $|q|<1$, is specified as

$$(x;q)_\infty := \prod_{n=1}^{\infty}\left(1-xq^{n-1}\right).$$

If $|xy|<1$, theta function in Ramanujan's general form is declared as follows:

$$f(x,y) := \sum_{n=-\infty}^{\infty} x^{n(n+1)/22}y^{n(n-1)/2}.$$

By Jacobi's triple product identity [[4], p.35], we have

$$f(x,y) = (-x,-y,xy;xy)_\infty .$$

The essential special cases of $f(x,y)$ [4, p.36], are

$$\phi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty^2$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{q^2;q^2)_\infty^2}{(q;q^2)_\infty}$$

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty.$$

For any complex number τ it is very clear that if $q = e^{2\pi i\tau}$ then $f(-q) = e^{-\pi i\tau/12}\eta(\tau)$, where $\eta(\tau)$ is the classical Dedekind η-function and is defined as

$$\eta(\tau) := q^{1/24} \prod_{n=1}^{\infty} (1-q^n) = e^{\pi i\tau/12} \prod_{n=1}^{\infty} (1-e^{2n\pi i\tau}),\ Im(\tau) > 0.$$

We use the notation

$$\chi(q) : -(-q;q^2)_\infty.$$

For convenience, we write $f(-q^n) = f_n$. The purpose of this article is to prove some of strange P-Q type modular equations of various degrees. Prior to pursue to prove these identities, we select initially to analyze some modular equations and theta-function identities which will be useful in future. A modular equation of degree n is an equation expressing α and β that is induced by

$$n\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\alpha\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right)} = \frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\beta\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\beta\right)},$$

where

$${}_2F_1(a,b,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} z^n \ |z| < 1$$

represents an ordinary hypergeometric function with

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}.$$

Then, we say that β is of n^{th} degree over α and call the ratio

$$m := \frac{z_1}{z_2},$$

where $z_1 =_2 F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right)$ and $z_2 =_2 F_1\left(\frac{1}{2},\frac{1}{2};1;\beta\right)$.

2. Main Results

Theorem 2.1. *If*

$$P = \frac{\phi^2(q^2)}{\phi(q)\phi(q^4)} \qquad \text{and} \qquad Q = \frac{\phi^2(q^4)}{\phi(q^2)\phi(q^8)}$$

then we have,

$$(PQ)^2 + \frac{2}{(PQ)^2} - 4\left(PQ + \frac{1}{PQ}\right) + \left(4\sqrt{PQ} - \frac{2}{\sqrt{PQ}}\right)\left(\sqrt{\frac{P}{Q}}\right)$$
$$+PQ\left(\frac{P}{Q} + \frac{Q}{P}\right) - 2P^3\sqrt{PQ}\left(\sqrt{\frac{P}{Q}} + \sqrt{\frac{Q}{P}}\right) + 3 = 0.$$

Proof. Let

$$A_n = \frac{\phi(q^n)}{\phi(q^{2n})}. \tag{1}$$

From (1) and together with the interpretation of P and Q, we have

$$P = \frac{A_2}{A_1} \text{ and } Q = \frac{A_4}{A_2}. \tag{2}$$

Also, from [10, Theorem 4.2], we have

$$A_1A_2 + \frac{2}{A_1A_2} = \frac{A_2}{A_1} + 2. \tag{3}$$

Also, from (2) and (3), we find that

$$\frac{A_1^2P}{\sqrt{2}} + \frac{\sqrt{2}}{A_1^2P} = \frac{1}{\sqrt{2}}(P+2). \tag{4}$$

On solving (4) for $A_1^2P/\sqrt{2}$. we deduce

$$\frac{A_1^2P}{\sqrt{2}} = \frac{k \pm \sqrt{k^2-4}}{2}, \tag{5}$$

where

$$k = \frac{1}{\sqrt{2}}(P+2).$$

The identity (5) implies that

$$\frac{\sqrt{2}}{A_1^2P} = \frac{k \pm \sqrt{l^2-4}}{2}. \tag{6}$$

On letting $q \to q^2$ in (3) and on solving for $A_2^2Q/\sqrt{2}$, we have

$$\frac{A_2^2Q}{\sqrt{2}} = \frac{l \mp \sqrt{l^2-4}}{2} \tag{7}$$

and

$$\frac{\sqrt{2}}{A_2^2Q} = \frac{l \mp \sqrt{l^2-4}}{2}, \tag{8}$$

where

$$l = \frac{1}{\sqrt{2}}(Q+2).$$

On multiplying (5) and (8), then employing (2) and after simplification, we obtain

$$\frac{4}{PQ} = kl \pm l\sqrt{k^2-4} \mp k\sqrt{l^2-4} - \sqrt{k^2-4}\sqrt{l^2-4}. \tag{9}$$

On multiplying (6) and (7), then employing (2), we obtain

$$4PQ = kl \mp l\sqrt{k^2-4} \mp k\sqrt{l^2-4} - \sqrt{k^2-4}\sqrt{l^2-4}. \tag{10}$$

Adding (9) and (10), and then streamlining, we obtain

$$2\left(PQ + \frac{1}{PQ}\right) - kl = \sqrt{k^2-4}\sqrt{l^2-4}.$$

Squaring on both sides, substituting the values of k and l and then streamlining, we obtain the desired result. □

Theorem 2.2. *If*

$$P = \frac{\phi^2(q^3)}{\phi(q)\phi(q^9)} \qquad \textit{and} \qquad Q = \frac{\phi^2(q^6)}{\phi(q)\phi(q^9)}$$

then we have,

$$3\left((PQ)^2 + \frac{1}{(PQ)^2}\right) - \left(10PQ + \frac{9}{PQ}\right) - (PQ)^3$$
$$-3(PQ-1)^2\left(\frac{P}{Q} + \frac{Q}{P}\right) + (PQ)^2\left(\left(\frac{P}{Q}\right)^2 + \left(\frac{Q}{P}\right)^2\right) + 12 = 0.$$

Proof. Let

$$A_n = \frac{\phi(q^n)}{\phi(q^{3n})}. \tag{11}$$

From (11) and together with the interpretation of P and Q, we have

$$P = \frac{A_3}{A_1} \text{ and } Q = \frac{A_9}{A_3}. \tag{12}$$

Also, from [10, Theorem 4.3], we have

$$A_1A_3 + \frac{2}{A_1A_3} = \left(\frac{A_3}{A_1}\right)^2 + 3. \tag{13}$$

Also, from (12) and (13), we find that

$$\frac{A_1^2P}{\sqrt{3}} + \frac{\sqrt{3}}{A_1^2P} = \frac{1}{\sqrt{3}}\left(P^2+3\right). \tag{14}$$

On solving (14) for $A_1^2P/\sqrt{3}$. we deduce

$$\frac{A_1^2P}{\sqrt{3}} = \frac{k \pm \sqrt{k^2-4}}{2}, \tag{15}$$

where

$$k = \frac{1}{\sqrt{3}}\left(P^2+3\right).$$

Also (15) implies

$$\frac{\sqrt{3}}{A_1^2P} = \frac{k \pm \sqrt{l^2-4}}{2}. \tag{16}$$

On changing $q \to q^2$ in (13) and in the similar manner, we deduce that

$$\frac{A_3^2 Q}{\sqrt{3}} = \frac{l \mp \sqrt{l^2-4}}{2} \tag{17}$$

and

$$\frac{\sqrt{3}}{A_3^2 Q} = \frac{l \mp \sqrt{l^2-4}}{2}, \tag{18}$$

where

$$l = \frac{1}{\sqrt{3}}\left(Q^3+3\right).$$

On multiplying (15) and (18), and employing (12) and after simplification, we obtain

$$\frac{4}{PQ} = kl \pm l\sqrt{k^2-4} \mp k\sqrt{l^2-4} - \sqrt{k^2-4}\sqrt{l^2-4}. \tag{19}$$

On multiplying (16) and (17), then employing (12), we obtain

$$4PQ = kl \mp l\sqrt{k^2-4} \mp k\sqrt{l^2-4} - \sqrt{k^2-4}\sqrt{l^2-4}. \tag{20}$$

Adding (19) and (20), and then streamlining, we obtain

$$2\left(PQ + \frac{1}{PQ}\right) - kl = \sqrt{k^2-4}\sqrt{l^2-4}.$$

Squaring on both sides, substituting the values of k and l and then streamlining, we obtain the desired result. □

Theorem 2.3. *If*

$$P = \frac{\psi(q)}{q\psi(q^9)} \qquad \text{and} \qquad Q = \frac{\psi(q^2)}{q^2\psi(q^{18})}$$

then we have,

$$\sqrt{2}\left(PQ + \frac{1}{PQ}\right)\left(\frac{\sqrt{2}P}{Q} + \frac{Q}{\sqrt{2}P} + \sqrt{2}\right) - 2\left(\frac{P}{Q} + \frac{Q}{P}\right)$$
$$+2\left(Q + \frac{1}{Q}\right) - \left(\frac{P^2}{Q} + \frac{Q}{P^2}\right) - \left(P^2Q + \frac{1}{P^2Q}\right) + 8 = 0.$$

Proof. Let

$$A_n = \frac{\psi(q^n)}{q^n\psi(q^{9n})}. \tag{21}$$

From (21) and together with the interpretation of P and Q, we have

$$P = \frac{A_2}{A_1} \text{ and } Q = \frac{A_4}{A_2}. \tag{22}$$

Also, from [6, Theorem 3.2], we have

$$\frac{A_1}{A_2} + \frac{A_2}{A_1} + 2 = \frac{3}{A_1} + A_1. \tag{23}$$

On using (22) and (23), we obtain

$$\frac{A_1}{\sqrt{3}} + \frac{\sqrt{3}}{A_1} = \frac{1}{\sqrt{3}}\left(P + \frac{1}{P} + 2\right). \tag{24}$$

On solving (24) for $A_1/\sqrt{3}$. we deduce

$$\frac{A_1}{\sqrt{3}} = \frac{k \pm \sqrt{k^2-4}}{2}, \tag{25}$$

where

$$k = \frac{1}{\sqrt{3}}\left(P + \frac{1}{P} + 2\right).$$

Also (25) implies

$$\frac{\sqrt{3}}{A_1} = \frac{k \pm \sqrt{l^2-4}}{2}. \tag{26}$$

On changing $q \to q^2$ in (23) and in the similar manner, we deduce that

$$\frac{A_2}{\sqrt{3}} = \frac{l \mp \sqrt{l^2-4}}{2} \tag{27}$$

and

$$\frac{\sqrt{3}}{A_2} = \frac{l \mp \sqrt{l^2-4}}{2}, \tag{28}$$

where

$$l = \frac{1}{\sqrt{3}}\left(Q + \frac{1}{Q} + 2\right).$$

On multiplying (25) and (28), and employing (22) and after simplification, we obtain

$$\frac{4}{P} = kl \pm l\sqrt{k^2-4} \mp k\sqrt{l^2-4} - \sqrt{k^2-4}\sqrt{l^2-4}. \tag{29}$$

On multiplying (26) and (27), then employing (22), we obtain

$$4P = kl \mp l\sqrt{k^2-4} \mp k\sqrt{l^2-4} - \sqrt{k^2-4}\sqrt{l^2-4}. \tag{30}$$

Adding (29) and (30), and then streamlining, we obtain

$$2\left(P + \frac{1}{P}\right) - kl = \sqrt{k^2-4}\sqrt{l^2-4}.$$

Squaring on both sides, substituting the values of k and l and then streamlining, we obtain the desired result. □

Theorem 2.4. *If*

$$P = q^{1/2}\frac{f_1 f_{26}}{f_2 f_{13}} \qquad \textit{and} \qquad Q = q\frac{f_2 f_{52}}{f_4 f_{26}},$$

then we have,

$$(PQ)^2 + \frac{1}{(PQ)^2} - 4\left(PQ + \frac{1}{PQ}\right)\left(\frac{P}{Q} + \frac{Q}{P}\right) - \left(PQ + \frac{1}{PQ}\right)$$
$$\times\left(\left(\frac{P}{Q}\right)^3 + \left(\frac{Q}{P}\right)^3\right) + 4\left(\left(\frac{P}{Q}\right)^2 + \left(\frac{Q}{P}\right)^2\right) + 18 = 0.$$

Proof. Let

$$A_n = \frac{f_n}{q^{\frac{n}{2}} f_{13n}}. \tag{31}$$

From (31) and together with the interpretation of P and Q, we have

$$P = \frac{A_1}{A_2} \text{ and } Q = \frac{A_2}{A_4}. \tag{32}$$

Also, from [10, Theorem 4.2], we have

$$A_1A_2 + \frac{13}{A_1A_2} = \left(\frac{A_2}{A_1}\right)^3 + \left(\frac{A_1}{A_2}\right)^3 - 4\left(\frac{A_1}{A_2}\right) - 4\left(\frac{A_2}{A_1}\right). \tag{33}$$

Also, from (32) and (33), we find that

$$\frac{A_1A_2}{\sqrt{13}} + \frac{\sqrt{13}}{A_1A_2} = \frac{1}{\sqrt{13}}\left(P^3 + \frac{1}{P^3} - 4P - \frac{4}{P}\right). \tag{34}$$

On solving (34) for $\frac{A_1A_2}{\sqrt{13}}$. we deduce

$$\frac{A_1A_2}{\sqrt{13}} = \frac{K_1 \pm \sqrt{k^2-4}}{2}, \tag{35}$$

where

$$k = \frac{1}{\sqrt{13}}\left(P^3 + \frac{1}{P^3} - 4P - \frac{4}{P}\right).$$

The identity (35) implies that

$$\frac{\sqrt{13}}{A_1A_2} = \frac{k \pm \sqrt{k^2-4}}{2}. \tag{36}$$

On letting $q \to q^2$ in (33) and on solving for $\frac{A_2A_4}{\sqrt{13}}$, we have

$$\frac{A_2A_4}{\sqrt{13}} = \frac{l \mp \sqrt{l^2-4}}{2} \tag{37}$$

and

$$\frac{\sqrt{13}}{A_2A_4} = \frac{l \mp \sqrt{l^2-4}}{2}, \tag{38}$$

where

$$l = \frac{1}{\sqrt{13}}\left(Q^3 + \frac{1}{Q^3} - 4Q - \frac{4}{Q}\right).$$

On multiplying (35) and (38), then employing (32) and after simplification, we obtain

$$\frac{4}{PQ} = kl \pm l\sqrt{k^2-4} \mp k\sqrt{l^2-4} - \sqrt{k^2-4}\sqrt{l^2-4}. \tag{39}$$

On multiplying (36) and (37), then employing (32), we obtain

$$4PQ = kl \mp l\sqrt{k^2-4} \mp k\sqrt{l^2-4} - \sqrt{k^2-4}\sqrt{l^2-4}. \tag{40}$$

Adding (39) and (40), and then streamlining, we obtain

$$2\left(PQ+\frac{1}{PQ}\right)-kl=-\sqrt{k^2-4}\sqrt{l^2-4}.$$

Squaring on both sides, substituting the values of k and l and then streamlining, we obtain the desired result. □

Theorem 2.5. *If*

$$P=q\frac{f_1f_{50}}{f_2f_{25}} \qquad \textit{and} \qquad Q=q^2\frac{f_2f_{100}}{f_4f_{50}}$$

then we have,

$$x_4+16x_3-24x_2+16x_1+4(2x_{1/2}-x_{3/2}-x_{7/2})y_{1/2}+4(x_{1/2}+x_{3/2}-2x_{5/2})y_{5/2} \\ -(x_1-x_3)y_3+16(x_1+x_3)y_2+18=0,$$

where

$$x_n=(PQ)^n+\frac{1}{(PQ)^n} \qquad \textit{and} \qquad y_n=\left(\frac{P}{Q}\right)^n+\left(\frac{Q}{P}\right)^n.$$

Proof. Let

$$A_n=\frac{f_n}{q^nf_{25n}}. \tag{41}$$

From (41) and together with the interpretation of P and Q, we have

$$P=\frac{A_1}{A_2} \text{ and } Q=\frac{A_2}{A_4}. \tag{42}$$

Also, from [10, Theorem4.2], we have

$$A_1A_2+\frac{25}{A_1A_2}=\left(\frac{A_2}{A_1}\right)^3+\left(\frac{A_1}{A_2}\right)^3-4\left(\frac{A_1}{A_2}\right)^2-4\left(\frac{A_2}{A_1}\right)^2. \tag{43}$$

Also, from (42) and (43), we find that

$$\frac{A_1A_2}{5}+\frac{5}{A_1A_2}=\frac{1}{5}\left(P^3+\frac{1}{P^3}-4P^2-\frac{4}{P^2}\right). \tag{44}$$

On solving (44) for $\frac{A_1A_2}{5}$. we deduce

$$\frac{A_1A_2}{5}=\frac{k\pm\sqrt{k^2-4}}{2}, \tag{45}$$

where

$$k=\frac{1}{5}\left(P^3+\frac{1}{P^3}-4P^2-\frac{4}{P^2}\right).$$

The identity (45) implies that

$$\frac{5}{A_1A_2}=\frac{k\pm\sqrt{k^2-4}}{2}. \tag{46}$$

On letting $q\to q^2$ in (33) and on solving for $\frac{A_2A_4}{5}$, we have

$$\frac{A_2A_4}{5}=\frac{l\mp\sqrt{l^2-4}}{2} \tag{47}$$

and

$$\frac{5}{A_2A_4} = \frac{l \mp \sqrt{l^2-4}}{2}, \tag{48}$$

where

$$l = \frac{1}{5}\left(Q^3 + \frac{1}{Q^3} - 4Q^2 - \frac{4}{Q^2}\right).$$

On multiplying (45) and (48), then employing (42) and after simplification, we obtain

$$4PQ = kl \mp l\sqrt{k^2-4} \mp k\sqrt{l^2-4} - \sqrt{k^2-4}\sqrt{l^2-4}. \tag{49}$$

On multiplying (46) and (47), then employing (42), we obtain

$$\frac{4}{PQ} = kl \pm l\sqrt{k^2-4} \mp k\sqrt{l^2-4} - \sqrt{k^2-4}\sqrt{l^2-4}. \tag{50}$$

Adding (49) and (50), and then streamlining, we obtain

$$2\left(PQ + \frac{1}{PQ}\right) = kl - \sqrt{k^2-4}\sqrt{l^2-4}.$$

Squaring on both sides and then streamlining, we obtain

$$(PQ)^2 + \frac{1}{(PQ)^2} - kl\left(PQ + \frac{1}{PQ}\right) + k^2 + l^2 - 2 = 0.$$

Finally, on substituting the values of k and l in the atop identity and further streamlining, we have the desired result. □

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